Exam — Functional Analysis (WBMA033-05), University of Groningen

Friday, April 8, 2022, 08.30h–10.30h CEST

Instructions

1. Except for the official *cheat sheet*, the use of books or notes is not allowed.
2. Justify all your answers.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.
4. Write both your last name and student number on the answer sheets.

Problem 1 ($3 \times 5 = 15$ points)

Let $\{\alpha_n\} \subset \mathbb{R}$ satisfy $0 < \inf_{n\in\mathbb{N}} \alpha_n \leq \sup_{n\in\mathbb{N}} \alpha_n < \infty$. For each $x \in \ell^1$, set $|||x||| = \sum_{n\in\mathbb{N}} \alpha_n |x_n|$.

(a) Show that $|||\cdot|||$ is a norm on ℓ^1.

(b) Prove that ℓ^1, with the norm $|||\cdot|||$, is a Banach space. Hint: There is a 1-line proof.

(c) Show that is ℓ^1, with the norm $|||\cdot|||$, is not a Banach space if $a_n > 0$ $\forall n$ but $\inf_{n\in\mathbb{N}} \alpha_n = 0$.

Problem 2 (10 points)

The Hahn-Banach Separation Theorem states that, in a normed space, any nonempty closed convex set not containing the origin can be strictly separated from the origin by a hyperplane. State this result in proper mathematical terms, assuming the underlying space is a real Hilbert space, and give a simple proof, based on the existence of orthogonal projections.

Problem 3 (15 points)

Let X be reflexive, and let $K \subset X$ be nonempty, closed and convex. Show that, for each $x \in X$, there is $\bar{y} \in K$ such that $\|x - \bar{y}\| = \operatorname{dist}(x, K) := \inf_{y\in K} \|x - y\|$.

Problem 4 (10 points)

Let X, Y be Banach spaces. Can a linear operator $T : X \to Y$ be compact and surjective?

Problem 5 ($8 \times 5 = 40$ points)

Let H be a complex Hilbert space with $\dim(H) \geq 2$, and let $y, z \in H \setminus \{0\}$. Define $T : H \to H$ by $Tx = \langle x, y\rangle z$ for $x \in H$.

(a) Compute $\|T\|$.

(b) Prove that T is compact.

(c) Determine T^*.

(d) Show that T is normal if, and only if, there is $c \in \mathbb{C}$ such that $z = cy$.

(e) Show that T is selfadjoint if, and only if, there is $c \in \mathbb{R}$ such that $z = cy$.

(f) Find $a, b \in \mathbb{C}$ (possibly depending on λ, y, z) such that $(T - \lambda)^{-1} = a + bT$ for all but two values of $\lambda \in \mathbb{C}$. Hint: $T^2 = \langle z, y\rangle T$.

(g) Determine all the eigenvalues of T.

(h) Determine $\rho(T)$ and $\sigma(T)$.